

Toroidal Compactification in String Theory from Chern-Simons Theory

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ABSTRACT

A detailed study of the charge spectrum of three dimensional Abelian Topological Massive Gauge Theory (TMGT) is given. When this theory is defined on a manifold with two disconnected boundaries there are induced chiral Conformal Field Theories (CFT's) on the boundaries which can be interpreted as the left and right sectors of closed strings. We show that Narain constraints on toroidal compactification (integer, even, self-dual momentum lattice) have a natural interpretation in purely three dimensional terms. This is an important result which is necessary to construct toroidal compactification and heterotic string from Topological Membrane (TM) approach to string theory. We also derive the block structure of $c = 1$ Rational Conformal Field Theory (RCFT) from the three dimensional gauge theory.

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1 Introduction

Inherent to all known perturbative string theories are the two dimensional conformal field theories. Moreover, keeping in mind the fact that a given string theory can be considered as a two dimensional sigma model (quantum field theory) which has spacetime as its target space, it is hard to over emphasize the importance of $2D$ conformal field theories. Most of the claimed beauties of string theories undoubtedly reside in the fact that one has free and conformal theories on the worldsheet. For the closed string case, which we exclusively work within this paper, an important feature of the the worldsheet theory is that left and right moving degrees of freedom decouple from each other. One can then formulate the theory by considering two chiral conformal field theories, holomorphic and anti-holomorphic, living on the same two dimensional surface Σ . The features of the non-chiral CFT, and the corresponding string theory, depend on how one combines the holomorphic and anti-holomorphic theories. This part of the story is well known. In this paper we continue to develop the idea of Topological Membrane approach to String Theory which was first set forward in [1, 2] and worked out in many subsequent papers, where the left and right moving sectors of closed strings live on two different surfaces Σ_L and Σ_R . In other words, we are giving a finite physical thickness (be it spatial or temporal) to the string worldsheet which becomes now a three-manifold.

Topological Membrane idea is based on the results of Witten [3] and Moore and Seiberg [4] (see also [5]), Chern-Simons theory defined on a three-manifold with a boundary is equivalent to $2D$ conformal field theory, gauged chiral WZWN model, living on the boundary. This is essentially the first example of a theory where the currently popular holographic principle works [6]. The chain of reasoning is as follows: the $3D$ bulk theory is equivalent to the boundary conformal theory and the later underlines a $10D$ string theory. So the hope is that at the very fundamental level of string theory lies a three dimensional quantum field theory. Although it is rather speculative at this stage this formulation might provide us with an off-shell formulation of string theory as well as new relations between different types of strings from a more fundamental theory.

The actual story is not as simple as it is laid above. Chern-Simons theory in the bulk is not enough to reproduce String Theory. See [2] for more complete details. Besides the fact that we must induce all aspects (moduli, Liouville fields, gravitational dressing, etc.) of $2d$ gravity [7] there is a problem of obtaining the spectrum of toroidal compactification [8] and heterotic string [9, 10] from the original pure Chern-Simons theory [3, 4]. This is due to the impossibility to have different right and left charge spectrum and to induce holomorphic

CFT in one boundary without inducing anything in the other. This problem is solved by adding a Maxwell term in the action and considering CS as the low energy limit of TMGT [11]. In this way monopoles emerge in the theory and induce new processes [12] which lead to local charge non conservation that permit a right charge to have a different value than a left charge. The important point is that bulk effects (monopoles) allow us to have winding modes in the boundary theory. Imposing suitable boundary conditions [13, 14] one can construct heterotic strings in this framework.

With the Maxwell term the bulk theory of course becomes non-topological and the holographic principle partially works. The spectrum of the Topological Massive Gauge Theory has a mass gap between its excited states and the degenerate ground state. But still the Hilbert subspace of the ground state of Abelian TMGT is exactly equivalent to the Hilbert space of the full CS theory. Moreover in order to define the quantum CS theory the Maxwell term has to be introduced as an ultraviolet regulator, which is set to zero after renormalization [15]. So in any case one needs the Maxwell term.

In [16] the charge non conservation induced by monopoles was discussed and these ideas were applied to T-duality and Mirror Symmetry in [17]. Nevertheless the problem of building the correct left-right spectrum has never been properly solved. The main goal of this paper is to determine the charge lattice structure allowed by TMGT. Essentially we show that, upon imposing suitable boundary conditions, the theory demands a charge lattice which is exactly of the form of string theory momentum lattice. From the non-perturbative dynamics of three dimensional gauge theory we derive the Narain lattice spectrum. We consider here both bosonic and heterotic string theories.

In section 2 we start with a short review of 3D TMGT and some related topics necessary for the present work, namely some results in Conformal Theories and String Theory. Next we move on to the main part of this paper, section 3, a model which describes the dynamics of charges propagating in the 3D bulk theory is built. We arrive to some well know results in string theory but derived purely from the dynamics of the bulk 3D theory. Namely the mass spectrum of toroidally compactified closed string theory emerges. In section 4 we study a relevant issue of gluing both 2D boundaries of the 3D manifold in order to get a single conformal field theory in the boundary. In section 5 the underlying structure of conformal block structure of the $c = 1$ compactified bosonic RCFT and the corresponding fusion rules are found as a result of monopole-instanton induced interactions in the bulk. Finally in section 6 the spectrum of heterotic string and possible backgrounds are rederived in the light of these new results.

2 A Short Review on TMGT

There are several ways to derive CFT from CS theory. We choose the path integral approach first suggested by Ogura [18] (see also [19]). In this section we review some features of the topologically massive gauge theory defined on a three dimensional flat manifold with a boundary. In order to clarify some points we present some arguments derived from canonical formalism. We present a list of the induced chiral boundary conformal field theories and we carefully analyze the Gauss law structure of the gauge theory with compact gauge group $U(1)$ given by the action

$$S = \int_{\mathcal{M}} d^2z dt \left[-\frac{\sqrt{-g(3)}}{4\gamma} F^{\mu\nu} F_{\mu\nu} + \frac{k}{8\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right] \quad (1)$$

where $\mathcal{M} = \Sigma \times [0, 1]$ and Σ is a 2 dimensional compact Euclidean manifold with a complex structure denoted by (z, \bar{z}) . The time-like coordinate takes values in the compact domain $t \in [0, 1]$. The indices run over $\mu = 0, i$ with $i = z, \bar{z}$.

Under an infinitesimal variation of the fields $A \rightarrow A + \delta A$ the action changes by

$$\delta S = \int_M \left(\frac{\sqrt{-g(3)}}{\gamma} \partial_\mu F^{\mu\nu} + \frac{k}{4\pi} \epsilon^{\mu\lambda\nu} \partial_\mu A_\lambda \right) \delta A_\nu - \left[\int_\Sigma \Pi^i \delta A_i \right]_{t=0}^{t=1} \quad (2)$$

where

$$\Pi^i = \frac{1}{\gamma} F^{0i} - \frac{k}{8\pi} \tilde{\epsilon}^{ij} A_j \quad (3)$$

is the canonical momenta conjugate to A_i . Note that the $2d$ antisymmetric tensor $\tilde{\epsilon}^{ij}$ is induced by the $3d$ antisymmetric tensor and metric

$$\tilde{\epsilon}^{ij} = \frac{-\epsilon^{0ij}}{\sqrt{-g(3)}} \quad (4)$$

When referring to the usual $2d$ antisymmetric tensor we use the notation ϵ^{ij} without the tilde.

In order the theory to have classical extrema it is necessary to impose suitable *boundary conditions* for which the second term in the variation of the action vanishes. Let us assume that the boundary of \mathcal{M} has two pieces, which are $\Sigma_{(t=0)} = \Sigma_L$ and $\Sigma_{(t=1)} = \Sigma_R$. L and R denote left and right. On each of the boundaries, up to gauge transformations, one can fix one or both fields A_z and $A_{\bar{z}}$. In doing so one should add an appropriate boundary action $S_B = S_{BL} + S_{BR}$ such that the new action $S + S_B$ has no boundary variation, hence well defined classical extrema. Note that upon canonical quantization we have to impose the corresponding equal time commutation relations of Π and A

$$[\Pi^i(\mathbf{z}), A^{\bar{j}}(\mathbf{z}')] = g^{i\bar{j}} \delta^{(2)}(\mathbf{z} - \mathbf{z}') \quad (5)$$

Our convention for the metric is $g^{z\bar{z}} = g^{\bar{z}z} = 2$. So fixing A^z we fix $\Pi^{\bar{z}}$ as well and the same holds for the $A^{\bar{z}}$ and Π^z components.

On each components of Σ the possible choices of boundary conditions and the boundary actions are

boundary conditions	left bound. action	right bound. action	
$N.$	$\delta A_z = \delta A_{\bar{z}} = 0$	$S_{BL} = 0$	$S_{BR} = 0$
$C.$	$\delta A_{\bar{z}} = 0$	$S_{BL} = - \int_{\Sigma_R} \Pi^z A_z$	$S_{BR} = \int_{\Sigma_L} \Pi^z A_z$
$\bar{C}.$	$\delta A_z = 0$	$S_{BL} = - \int_{\Sigma_R} \Pi^{\bar{z}} A_{\bar{z}}$	$S_{BR} = \int_{\Sigma_L} \Pi^{\bar{z}} A_{\bar{z}}$

(6)

There are nine allowed choices: NN , NC , CC , $C\bar{C}$, and so on. The first letter denotes the type of left boundary conditions and the second one the right type. The N boundary condition stands for *Non-Conformal* or *Non-Dynamical*, C stands for *Conformal* and \bar{C} for *anti-Conformal* and are related to the kind of CFT we obtain in the boundaries when we choose them. Note the importance of the F^2 term, it gives the theory four independent canonical coordinates, opposed to the two of the pure Chern-Simons theory (where one of the A 's is canonically conjugate to the other). This fact allow us to fix one of the A 's and corresponding Π without fixing the other two allowing us to have different boundary conditions in opposite boundaries. For further details we refer the reader to [13, 14]. This topic will be addressed again in section 4.

By Faddeev-Popov procedure we can fix the gauge $A_\mu = \bar{A}_\mu + \partial_\mu \Lambda$. The action and path integral factorizes as

$$Z = \int \mathcal{D}\bar{\mathbf{A}} \Delta_{FP} \delta(F(\bar{\mathbf{A}})) e^{i(\bar{S} + \bar{S}_B)[\bar{\mathbf{A}}]} \int \mathcal{D}\lambda \mathcal{D}\chi e^{i\delta\bar{S}_L[\chi] + i\delta\bar{S}_R[\lambda]} \quad (7)$$

where in the boundaries the gauge parameters $\Lambda(t=0) = \chi$ and $\Lambda(t=1) = \lambda$ become dynamical degrees of freedom which decouple from the bulk theory.

Now let us find the allowed charges in this theory. The compactness of the gauge group has to be taken into account and it turns out to be of fundamental importance. Defining the electric and magnetic fields as

$$\begin{aligned} E^i &= \frac{1}{\gamma} F^{0i} = \Pi^i + \frac{k}{8\pi} \tilde{\epsilon}^{ij} A_j \\ B &= \epsilon^{ij} \partial_i A_j \end{aligned} \quad (8)$$

(For the sake of notational simplicity we have omitted the complex conjugation on the indices) The commutation relations read

$$\begin{aligned}[E^i(\mathbf{z}), E^j(\mathbf{z}')] &= -i\frac{k}{4\pi}\epsilon^{ij}\delta^{(2)}(\mathbf{z} - \mathbf{z}') \\ [E^i(\mathbf{z}), B(\mathbf{z}')] &= -i\epsilon^{ij}\partial_j\delta^{(2)}(\mathbf{z} - \mathbf{z}')\end{aligned}\tag{9}$$

If there is an external charge the Gauss law is

$$\partial_i E^i + \frac{k}{4\pi}B = \rho_0\tag{10}$$

In the quantum theory this equation needs to be satisfied by the physical states. So following [16] we can define the generator of time independent gauge transformations U

$$U = \exp \left\{ i \int_{\Sigma} \Lambda(\mathbf{z}) \left(\partial_i E^i + \frac{k}{4\pi}B - \rho_0 \right) \right\}\tag{11}$$

Σ stands for a generic fixed time slice of \mathcal{M} . Since The gauge group is compact Λ is identified with an angle in the complex plane such that

$$\begin{aligned}\ln(z) &= \ln|z| + i(\Lambda(z) + 2\pi n) \\ \partial_i \Lambda(z) &= -\epsilon_{ij}\partial_j \ln|z|\end{aligned}\tag{12}$$

where the second equation follows from the Cauchy-Riemann equations. This last condition on Λ will restrict the physical Hilbert space in the compact theory [20]. Let us define a new operator

$$V(\mathbf{z}_0) = \exp \left\{ -i \int_{\Sigma} d^2z \left[\left(E^i + \frac{k}{4\pi}\epsilon^{ij}A_j \right) \epsilon^{ik}\partial_k \ln|\mathbf{z}_0 - \mathbf{z}| - \Lambda(\mathbf{z}_0 - \mathbf{z})\rho_0 \right] \right\}\tag{13}$$

The physical states of the theory must be gauge invariant (under U operator) as well eigenstates of this new local operator. Using the identity $\partial_k\partial_k \ln|z| = 2\pi\delta(z)$ and the commutation relations (9) for E and B we obtain

$$[B(\mathbf{z}), V^n(\mathbf{z}_0)] = 2\pi n V^n(\mathbf{z}_0) \delta^{(2)}(\mathbf{z} - \mathbf{z}')\tag{14}$$

This means that the operator V creates a pointlike magnetic vortex at \mathbf{z}_0 with magnetic flux

$$\int_{\Sigma} B = 2\pi n \quad n \in \mathbb{Z}\tag{15}$$

As stated before the F^2 term is fundamental for the existence of these tunneling processes that hold local charge non conservation, see [12, 20] for further details. Instantons in three dimensions are the monopoles in four dimensions. So in the rest of the paper we will make use of the terms instanton and monopole without any distinction.

Using the functional Schrödinger representation $\Pi^i = i\delta/\delta A_i$ and imposing the condition that phase acquired by a physical state under a gauge transformation be single valued we obtain the charge spectrum

$$q = m + \frac{k}{8\pi} \int_{\Sigma} B = m + \frac{k}{4} n \quad m, n \in \mathbb{Z} \quad (16)$$

As it can be seen from the above formula for a generic, non-integer, value of k the allowed charges are non-integers. In principle in the compact Abelian theory one expects that the charges are quantized as it happens in the Maxwell theory. But the existence of the Chern-Simons term changes this picture, the charges are quantized through m and n dependence but we consider k itself not quantized. As will be explained below this fact is only compatible (and demands) with the existence of a non compact gauge sector in the theory.

A charge q propagating in the bulk can interact with one monopole with flux (15) changing by an amount

$$\Delta q = \frac{k}{2} n \quad (17)$$

The path of a charge in the bulk can be thought as a Wilson line. The phase induced by the linking of two Wilson lines carrying charges q_1 and q_2 is

$$\langle W_{q_1} W_{q_2} \rangle = \exp \left\{ 2\pi i \frac{2}{k} q_1 q_2 l \right\} \quad (18)$$

where l is the linking number between the lines. The above computation was done in the limit of vanishing Maxwell term and with the assumption that there is no self-linking for the individual Wilson lines. The connection between the boundary CFT's and the bulk theory can be achieved by noting that the bulk gauge fields become pure gauges in the boundary and the Wilson lines in the boundary are none other than the vertex operators in Σ_L and Σ_R with momenta q

$$\begin{aligned} V_{R,q_R} &= \exp \left\{ -iq \int A_{\nu} dx^{\nu} \right\} \Big|_{\Sigma_R} = \exp \{ -iq_R \lambda \} \\ V_{L,q_L} &= \exp \left\{ -iq \int A_{\nu} dx^{\nu} \right\} \Big|_{\Sigma_L} = \exp \{ -iq_L \chi \} \end{aligned} \quad (19)$$

The conformal dimensions of these vertices is

$$\begin{aligned} \Delta_R &= \frac{q_R^2}{k} \\ \Delta_L &= \frac{q_L^2}{k} \end{aligned} \quad (20)$$

k appears because the induced action in the boundary is that of a chiral string action multiplied by k . To see the Wilson line and vertex operator correspondence let us obtain

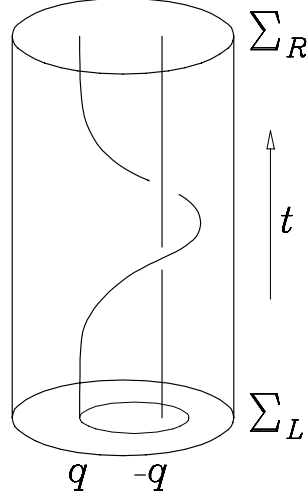


Figure 1: Two charges propagating through the bulk are represented as two Wilson lines, By a rotation of one charge around the other we induce one linking ($l = 1$) in the bulk.

the above result from the bulk theory. Consider two Wilson lines carrying charges q and $-q$ propagating from one boundary to the other corresponding to two vertex insertion in the boundary with momenta q and $-q$, see figure (1).

Two point correlation of these two vertices follows as usual

$$\begin{aligned}
 \langle V_{L,q}(\mathbf{z}_1) V_{L,-q}(\mathbf{z}_2) \rangle &= \frac{1}{\mathbf{z}_{12}^{2\Delta}} \\
 \langle V_{R,q}(\bar{\mathbf{z}}_1) V_{R,-q}(\bar{\mathbf{z}}_2) \rangle &= \frac{1}{\bar{\mathbf{z}}_{12}^{2\Delta}}
 \end{aligned} \tag{21}$$

A rotation of one charge (vertex) around the other in one boundary induces a phase of $-4\pi i\Delta$ in (21). In the bulk this rotation induces one linking of the Wilson lines ($l \rightarrow l+1$), so (18) gives an Aharonov-Bohm phase change of $-4\pi i q^2/k$. Identifying these two phases we conclude that (20) follows. Let us recall that in terms of the three dimensional theory Polyakov [21] pointed out that Δ is the transmuted spin of a charged particle which exists because of the interaction with Chern-Simons term. So we identify the conformal dimension of the boundary fields with the transmuted spin of the bulk charges. For later use let us generalize this result two three charges q_1 , q_2 and $q_3 = -q_1 - q_2$. We will always take the net charge to be zero. Three point correlation of three vertices reads

$$\langle V_{q_1}(\mathbf{z}_1) V_{q_2}(\mathbf{z}_2) V_{q_3}(\mathbf{z}_3) \rangle = \frac{1}{\mathbf{z}_{12}^{\Delta_1+\Delta_2-\Delta_3} \mathbf{z}_{13}^{\Delta_1+\Delta_3-\Delta_2} \mathbf{z}_{23}^{\Delta_2+\Delta_3-\Delta_1}} \tag{22}$$

where the charge-conformal dimension identifications are

$$\begin{aligned}\Delta_1 + \Delta_2 - \Delta_3 &= -\frac{2}{k}q_1q_2 \\ \Delta_1 + \Delta_3 - \Delta_2 &= \frac{2}{k}q_1(q_1 + q_2) \\ \Delta_2 + \Delta_3 - \Delta_1 &= \frac{2}{k}q_2(q_1 + q_2)\end{aligned}\tag{23}$$

The anti-holomorphic three point correlation follows trivially. It is clear that if any one of the vertices is rotated around one of the other two; the phase factor induced in the three-point function will be the same as (18) which comes from the linking of Wilson lines in the bulk. From the point of view of CFT the sum of conformal factors must be integer in order to have single valued OPE's. This result is going to be derived from the bulk theory in the next section independent of the boundary. By a simple argument let us show that we can only have integer conformal dimensions for each vertex. At the level of the two point correlation functions we can have two integer Δ 's or two half integer Δ 's. For three point functions it is necessary to have three integer Δ ' or two integer and one half integer Δ 's. One can then consider the four point function and decompose it to two pairs which are well separated from each other. In this case the conformal dimensions can be integers and half-integers. Suppose that at least two of the vertices have half-integer conformal dimensions. Then one vertex from a pair can be adiabatically moved to the other pair and a three point function with a total half-integer conformal dimension will be formed. But this is not permitted as argued above. For this reason we exclude the existence of vertices with half integer conformal dimension. The same arguments follow for the charges from the point of view of the $3d$ bulk theory.

Let us remember that we have two independent chiral conformal field theories on two different surfaces up to this point. In order to get a non-chiral CFT (as in string theory) we have to identify in some way the two boundaries. The most obvious way to define non-chiral vertex operators is

$$V_{q_R, q_L}(\mathbf{z}, \bar{\mathbf{z}}) = V_{R, q_R}(\mathbf{z})V_{L, q_L}(\bar{\mathbf{z}})\tag{24}$$

with the conformal dimension

$$\Delta_R + \Delta_L = \frac{1}{k}(q_R^2 + q_L^2)\tag{25}$$

Note that we have a little bit of difference in the nomenclature. In most of the literature Δ_R and Δ_L are called conformal weights and $\Delta_R - \Delta_L$ is called the spin and the sum (25) is called the conformal dimension. Here we call Δ_R and Δ_L conformal dimensions as well when we refer to the chiral vertices. From now on we will change our notation to $\Delta = \Delta_R$ and $\bar{\Delta} = \Delta_L$, the same for the charges.

For generic k and q 's, Δ in (20) and the sums in (23) are not integers. This is not a problem, actually is very welcome since it is simply the statement that we need something else in our theory, a new non compact gauge group sector. Or in terms of string theory, non compactified dimensions. So in addition to the action (1) with compact gauge group $U(1)$ we will consider the following action with non compact gauge group $U(1)^D$

$$S_D = \int_M d^2z dt \left[-\frac{1}{4\gamma'} f_M^{\mu\nu} f_{\mu\nu}^M + \frac{k'\delta_{MN}}{8\pi} \epsilon^{\mu\nu\lambda} a_\mu^M \partial_\nu a_\lambda^N \right] \quad (26)$$

where N and M run from 0 to $D-1$. We could take generally a coupling tensor K'_{MN} , for the purposes of this work is enough to consider the above $K'_{MN} = k'\delta_{MN}$. We will consider the actions (1) and (26) together. A generic (meaning that not necessarily the tachyon operators we considered before) non-chiral vertex of the boundary CFT is of the following form

$$\partial_z^s \zeta \partial_{\bar{z}}^{s'} \zeta \left(\prod_i \partial_z^{r_i} \Omega^M \right) \left(\prod_i \partial_{\bar{z}}^{r'_i} \Omega^M \right) \exp \left\{ -i \left[(q + \bar{q}) \zeta + p_M \Omega^M \right] \right\} \quad (27)$$

The fields ζ and Ω^M correspond to the gauge parameters of A and a^M respectively. Levels of the vertex operators are integers defined as

$$\begin{aligned} L &= s + \sum_i r_i \\ \bar{L} &= s' + \sum_i r'_i \end{aligned} \quad (28)$$

The exponential part may be represented as the bulk Wilson line propagating from boundary to boundary

$$W_D = \exp \left\{ -i \int [q A_\nu + p_M a_\nu^M] dx^\nu \right\} \quad (29)$$

The mass shell condition for the boundary vertex is

$$p_M p^M = -\text{mass}^2 \quad (30)$$

The mass spectrum in CFT's is built out of the allowed values for the conformal dimension of the fields or the operators. In particular the vertex operators (27) have conformal dimensions

$$\begin{aligned} \Delta &= \frac{q^2}{k} + \frac{p^2}{k'} + L \\ \bar{\Delta} &= \frac{\bar{q}^2}{k} + \frac{p^2}{k'} + \bar{L} \end{aligned} \quad (31)$$

Due to conformal invariance one has $\Delta = \bar{\Delta} = 1$. To get the usual String theory normalization one should replace $k' = 4/\alpha = k/R^2$ and take the average of both equations

(31)

$$\text{mass}^2 = -p^2 = \frac{(q^2 + \bar{q}^2)k'}{2k} + \frac{k'}{2}(L + \bar{L} - 2) = \frac{m^2}{R^2} + \frac{R^2 n^2}{\alpha^2} + \frac{2}{\alpha}(L + \bar{L} - 2) \quad (32)$$

Subtracting one equation from the other in (31) one gets

$$\frac{q^2 - \bar{q}^2}{k} + L - \bar{L} = nm + L - \bar{L} = 0 \quad (33)$$

Let us call (32) and (33) the mass shell condition and spin condition respectively. For further details see [22]. We used the explicit forms of the charges corresponding to momentum spectrum of string theory, in our normalization

$$\begin{aligned} q &= m + \frac{k}{4}n \\ \bar{q} &= m - \frac{k}{4}n \end{aligned} \quad (34)$$

But remember that the allowed charges in the theory are of the q -form. In principle \bar{q} is not related to q and it should be of the form $\bar{q} = m' + n'k/4$. As it will be explained in detail in the rest of the paper we have the following picture in mind. A charge q is inserted in one of the boundaries and it goes through the bulk interacting with the gauge fields until it reaches the other boundary. During its journey through the bulk its interactions with (in)finite many monopole-instantons induce a change of its charge $\Delta q = Nk/2$. The charge emerges in the other boundary as $\bar{q} = q + Nk/2 = m + (n + 2N)k/4$. If $N = -n$ we have $\bar{q} = m - nk/4$, this is what we are looking for. But why N must be $-n$? How do the monopoles know a priori they have to interact by this amount with the initial charge q ? In principle it could exist any other process holding $N \neq -n$ that would generate some charge \bar{q} leading us in disaster. We would get the pair (q, \bar{q}) with no correspondence with any physical sensible momentum pair (P_R, P_L) of string theory. So we have to show that the full $3d$ -amplitudes $q \rightarrow \bar{q}$ in the theory corresponding to unwanted processes $\bar{q} \neq m - nk/4$ are vanishing! This crucial property shows that seemingly independent chiral field theories on different boundaries are actually related when the non-perturbative excitations in the bulk are taken into account. This fact is closely related to considering the Maxwell Chern-Simons theory instead of the pure Chern-Simons and the presence in the bulk of monopole processes. In the next section we shall prove that the theory indeed has this property.

3 Propagation in the bulk

Our aim in this section is to investigate the changes that a charge undergo when it travels from one boundary to the other. As it will be shown the bulk theory imposes, independent of

what the boundary conditions are, restrictions for the charges. We are assuming irrational k in this section and in section 5 the case of rational k will be addressed.

We are dealing with a closed manifold \mathcal{M} with zero total charge on it. The instantons can induce a charge non-conservation locally only. So let us insert a pair of charges $Q_1 = -Q_2$ in one boundary, say Σ_L . They will travel through the bulk and emerge in the opposite boundary, Σ_R as two new charges $\bar{Q}_1 = -\bar{Q}_2$. In their paths the charges can interact with the bulk fields. Their paths can link but we assume there is no self-linking and the charges do not interact with each other.

Translating this to a more formal language. We have two open Wilson lines, each of them have their ends attached to different boundaries. The Wilson line represents the path of a charged particle. Leaving aside the perturbative interactions in the bulk, there are charge non-conserving interactions with the monopole-instantons. We will represent these interactions as insertions of instantons along the Wilson lines. Each instanton on the path of the particle changes the sign of the particle as mentioned before. So the physical picture is that we have a Wilson line on which the charge varies from point to point. But this variation is not random and it is induced by instantons. One can in principle consider chains of Wilson lines which is essentially the same picture. In between two insertions the two lines (minimal Wilson line segments with constant charges on them) can link to each other changing the correlator of the lines according to (18). So we assume that we can separate the insertions of instantons from the linkings of the lines. Or an other way to put is that insertions of instantons are done at the ends of the Wilson line segments.

A generic propagation of two such charges with j instanton interactions and $j + 1$ sets of linkings in the bulk is represented as a planar diagram in figure 2. The insertions are represented by crosses and the linkings by horizontal lines. The full physical picture is obtained by taking the limit $j \rightarrow \infty$ which allows infinitely many instanton insertions on a line. Since we assume global charge conservation, $\bar{Q}_1 - Q_1 = Q_2 - \bar{Q}_2$ and the upper charges are the result of all the monopole contributions $\bar{Q}_1 = Q_1 + \sum k/2N_i$. So we have to impose the condition

$$\sum_{i=1}^j N_i = N = - \sum_{i=1}^j N'_i \quad (35)$$

Each pair of charges can be formally interpreted as a quantum state $(Q_1, -Q_1)$ and its propagation from one boundary to the other boundary can be considered as a time evolution to the state $(\bar{Q}_1, -\bar{Q}_2)$. Then for each initial Q_1 and final \bar{Q}_1 we can built a transition

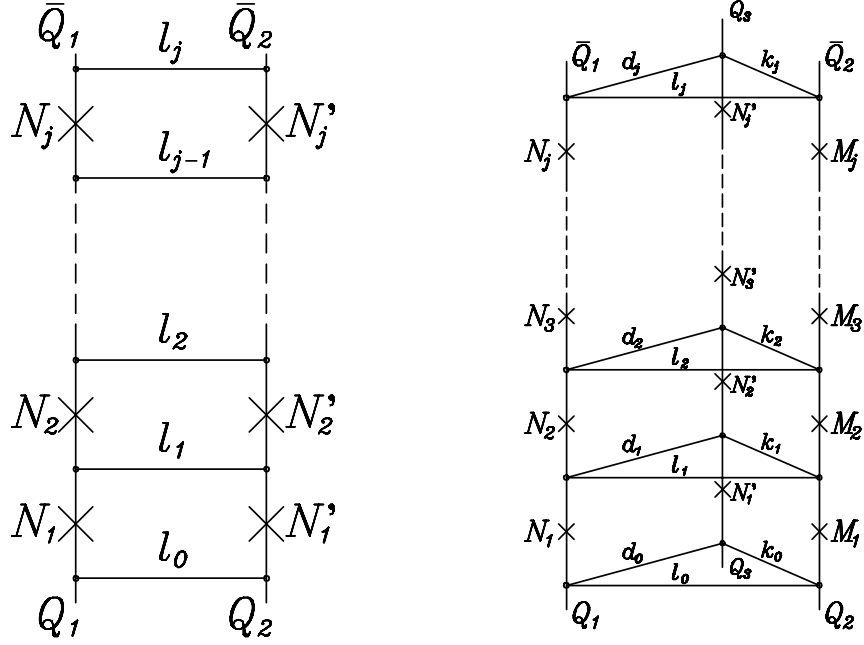


Figure 2: Two and three charges propagation through the bulk. The N 's, M 's, and N' 's represent the flux of instantons and the l 's, d 's and k 's the linking number of the Wilson lines between insertions.

coefficient $C[Q_1, N]$ which is to be interpreted as the square root of probability.

$$(Q_1, -Q_1) \rightarrow C[Q_1, N] (Q_1 + \frac{k}{2}N, -Q_1 - \frac{k}{2}N) \quad (36)$$

where we already make use of condition (35). C is a function of Q_1 and N only but in order to find it we need to take into account all the possible processes in the bulk, monopole interactions and linkings. To be able to extract some information about the boundary let us insert a new variable l , which denotes the total linking number. Up to normalization factors C can be written as

$$C[Q_1, N] = \sum_{\tilde{l}=-\infty}^{\infty} \int dl \delta(\tilde{l} - l) c[Q_1, N, l] \quad (37)$$

with the definition

$$l = l_0 + \sum_{i=1}^j l_i \quad (38)$$

It is now about time to find how the infinitely many bulk processes contribute to this coefficient. We know that interacting with the monopoles labeled by i the pair of charges will go through the following transition $(q, q') \rightarrow (q + k/2N_i, q' + k/2N'_i)$ according to (17).

Between each monopole insertions the linking will induce a phase change of $4\pi i q q' l_i / k$ according to (18). The total change of the charges is already taken in account through N dependence on C . But the processes that we are summing over must be selected in order to fulfill the requirement that the total monopole contribution is N and that the total linking number is l . The coefficient c is then the sum of the phases

$$c[Q_1, N, l] = \sum_{N_a, N'_b} \sum_{l_i, l_0} \exp\{2\pi i \Phi[N_a, N'_b, l_i, l_0]\} \quad (39)$$

Where the sums over N_a and N'_b stand for all the allowed configurations obeying (35) and the sum over l_i and l_0 for the ones obeying (38). The phase, Φ , for each process is simply the sum of the several phases induced at each step i (see figure 2) along the propagation in the bulk

$$\Phi = \frac{2}{k} \left\{ -Q_1^2 l_0 + \sum_{i=1}^j \left(Q_1 + \frac{k}{2} \sum_{b=1}^i N_b \right) \left(-Q_1 + \frac{k}{2} \sum_{a=1}^i N'_a \right) l_i \right\} \quad (40)$$

Writing the last term of the sum in i using (35), i.e. for $i = j$, and expanding the products we obtain

$$\begin{aligned} \Phi = & \frac{2}{k} \left\{ -Q_1^2 l_0 - Q_1^2 \sum_{i=1}^j l_i + \left(Q_1^2 - \left(Q_1 + \frac{k}{2} N \right)^2 \right) l_j \right. \\ & \left. + \frac{k}{2} \sum_{i=1}^{j-1} \left(Q_1 \sum_{a=1}^i (N'_a - N_a) + \frac{k}{2} \sum_{a=1}^i N'_a \sum_{b=1}^i N_b \right) l_i \right\} \end{aligned} \quad (41)$$

Note that the phase is no longer N_j dependent, we replaced it by N dependence. We further need to get the l dependence of c . In the following discussion we are going to investigate the cases for which the coefficient is non-vanishing. Bearing in mind that we are dealing with formal series (which are divergent in general) different ways of factorizing the sums is safer. So we will depict three different factorizations below, c will factorize into a l dependent phase and an independent phase factor. Unless otherwise stated from now on the index i will run from 1 to $j-1$.

The most obvious way to factorize the sum is to eliminate $l_0 = l - \sum l_i - l_j$. Then (39) factorizes as

$$\begin{aligned} c[Q_1, N, l] = & \exp \left\{ -2\pi i \frac{2}{k} Q_1^2 l \right\} \times \\ & \times \sum_{N_a=-\infty}^{\infty} \sum_{N'_b=-\infty}^{\infty} \prod_{i=1}^{j-1} \sum_{l_i=-\infty}^{\infty} \sum_{l_j=-\infty}^{\infty} \exp\{2\pi i \Phi'_0[N_a, N_b, l_i, l_j]\} \end{aligned} \quad (42)$$

The indices a and b run from 1 to $j - 1$. Since the phase Φ'_0 is the sum of several phase changes ϕ we can rewrite the second factor as

$$\sum_{N_a, N'_b} \left(\prod_{i=1}^{j-1} \sum_{l_i=-\infty}^{\infty} \exp\{2\pi i \phi_i l_i\} \right) \left(\sum_{l_j=-\infty}^{\infty} \exp\{2\pi i \phi_j l_j\} \right) \quad (43)$$

where

$$\begin{aligned} \phi_i &= Q_1 \sum_{a=1}^i (N'_a - N_a) + \frac{k}{2} \sum_{a=1}^i N'_a \sum_{b=1}^i N_b \\ \phi_j &= \frac{2}{k} \left\{ Q_1^2 - \left(Q_1 + \frac{k}{2} N \right)^2 \right\} \end{aligned} \quad (44)$$

Now we can consider each of the sums over l_j and the l_i 's independently. If one of them is zero c will be vanishing. Note that we didn't worry too much about normalization factors but each of these phase sums must be normalized to a number between 0 and 1 in order that one has the interpretation of C^2 as a probability. Further, if we take the limit of $j \rightarrow \infty$ these sums will become integrals. We can investigate if they are zero or not by using the identity

$$\sum_{q=-\infty}^{\infty} \exp\{2\pi i \phi q\} = \sum_{p=-\infty}^{\infty} \delta(\phi - p) \quad (45)$$

The sum over delta functions is zero if ϕ is not an integer. The conclusion is then that, for every i , ϕ_i must be an integer. Summing over l_j we obtain the restriction

$$\frac{2}{k} \left\{ Q_1^2 - \left(Q_1 + \frac{k}{2} N \right)^2 \right\} \in \mathbb{Z} \quad (46)$$

Replacing Q_1 by its form (16), expanding and getting rid of the even integer term $2mN$ which does not change anything, the former condition reads

$$\frac{k}{2} N (n + N) \in \mathbb{Z} \quad (47)$$

For irrational k the only two solutions are

$$N = 0 \quad (48)$$

$$N = -n \quad (49)$$

The physical meaning of these results will be given a little later. The remaining conditions in the ϕ_i 's will allow us to build the intermediate processes, these will be addressed in the end of this section.

Now an other way to carry out the sums is to eliminate $l_j = l - l_0 - \sum l_i$. Then (39) factorizes as

$$c[Q_1, N, l] = \exp \left\{ -2\pi i \frac{2}{k} \left(Q_1 + \frac{k}{2} N \right)^2 l \right\} \times \sum_{N_a=-\infty}^{\infty} \sum_{N'_b=-\infty}^{\infty} \prod_{i=1}^{j-1} \sum_{l_i=-\infty}^{\infty} \sum_{l_0=-\infty}^{\infty} \exp \{ 2\pi i \Phi'_j[N_a, N_b, l_i, l_0] \} \quad (50)$$

Performing the sum over l_0 we obtain the same conditions (46) and (47) on N . There are extra $2N$ and N^2 factors but they do not change the coefficient c . We can check this explicitly by replacing Q_1 by its form (16). Apart from an irrelevant factor of $4\pi i m N$ one gets the same result as for Φ'_0 .

An other way of factorization is as the following. Write (41) in terms of l^\pm

$$l_0^+ = l_0 + l_j \quad l_0^- = l_0 - l_j \quad (51)$$

and take into account that $l_0^+ = l - \sum l_i$. Then (39) factorizes as

$$c[Q_1, N, l] = \exp \left\{ -2\pi i \frac{2}{k} \left(Q_1^2 + \left(Q_1 + \frac{k}{2} N \right)^2 \right) \frac{l}{2} \right\} \times \sum_{N_a=-\infty}^{\infty} \sum_{N'_b=-\infty}^{\infty} \sum_{l_i=-\infty}^{\infty} \sum_{l_0^-=-\infty}^{\infty} \exp \{ 2\pi i \Phi'_+[N_a, N_b, l_i, l_0^-] \} \quad (52)$$

Performing the sum over l_0^- we obtain a different condition

$$\frac{2}{k} \left\{ Q_1^2 - \left(Q_1 + \frac{k}{2} N \right)^2 \right\} \in 2\mathbb{Z} \quad (53)$$

Expanding Q_1 this gets to

$$\frac{k}{2} N (n + N) \in 2\mathbb{Z} \quad (54)$$

For generic $k \in \mathbb{R}$ we obtain the same solutions as (46) and apparently this condition turns out to hold the same results of (47). It will become clear that this last result is different from the previous two cases. As before the extra N and $N^2/2$ factors do not change the coefficient.

It is now time to explain the physical implications of the three previous results. The phase factors in front of l are none other than the conformal dimensions (20) of the vertex operators inserted in the boundary CFT's. When we eliminated the sum over l_0 we obtained (42). Extracting the l phase factor we get

$$\frac{2}{k} Q_1^2 = 2\Delta = \Delta_1 + \Delta_2 \quad (55)$$

Upon the elimination of l_j sum we obtained the factorization (50). The phase factor reads

$$\frac{2}{k}\bar{Q}_1^2 = 2\bar{\Delta} = \bar{\Delta}_1 + \bar{\Delta}_2 \quad (56)$$

And for the (52) factorization where we eliminated l_0^+ sum, we obtain

$$\frac{1}{k}(Q_1^2 + \bar{Q}_1^2) = \Delta + \bar{\Delta} \quad (57)$$

Note that we cannot extract any other factors, If we try to eliminate any other combination of l_0 and l_j we would get $c = 0$. Eliminating some combination of l_i 's we will always end up extracting one of the above factors or get again $c = 0$. The solutions for the monopole interactions (63) and (65) computed below assure this result.

In order C not to be zero it is necessary to get integer conformal dimensions as stated in section 2. Consider then adding extra non compact gauge fields. One Wilson line (29) carries now charges from all the $U(1)$'s. Summing over l for the three previous cases we obtain the conditions

$$\begin{aligned} r &= 2\left(\Delta + \frac{p^2}{k'}\right) \in \mathbb{Z} \\ s &= 2\left(\bar{\Delta} + \frac{p^2}{k'}\right) \in \mathbb{Z} \\ t &= \Delta + \bar{\Delta} + \frac{2p^2}{k'} \in \mathbb{Z} \end{aligned} \quad (58)$$

Subtracting the second equation from the first we obtain

$$r - s = 2(\Delta - \bar{\Delta}) = \frac{2}{k}(Q_1^2 - \bar{Q}_1^2) = -2mN \quad (59)$$

where $N = 0, -n$ according to the allowed solutions (48) and (49). Averaging the first two conditions we get the third one as long as we identify

$$t = (r + s)/2 \quad (60)$$

According to the discussion in section 2 r and s have to be even in order the full conformal dimension to be integer. Making the identifications

$$\begin{aligned} \frac{r}{2} &= 1 - L \\ \frac{s}{2} &= 1 - \bar{L} \end{aligned} \quad (61)$$

we retrieve the mass shell condition (32). And (59) becomes the spin condition (33).

Below we explore the possible intermediate processes by performing the sums over each of the l_i 's. We obtain a chain of conditions $\phi_i \in \mathbb{Z}$ which allow us to build the full set of possible diagrams contributing to the transitions. Starting with $i = 1$ we find

$$n(N'_1 - N_1) + 2N_1N'_1 = 0 \quad (62)$$

which has two possible solutions

$$\begin{array}{lcl} a & N_1 & = 0 = N'_1 \\ b & N_1 & = -n = -N'_1 \end{array} \quad (63)$$

that we name a and b .

For $i = 2$ we have

$$n(N'_1 - N_1) + n(N'_2 - N_2) + 2N_1N'_1 + 2N_2N'_2 + 2N_1N'_2 + 2N_2N'_1 = 0 \quad (64)$$

Choosing a for N_1 and N'_1 we have again the same previous solutions for N_2 and N'_2 . Choosing b we obtain again a or a new solution

$$\tilde{b} \quad N_2 = n = -N'_2 \quad (65)$$

that we name \tilde{b} . If we choose \tilde{b} we will find for $i = 3$ an a or b solution. So no new solutions emerge.

Note that a solution a will reduce the diagram to one of type $i - 1$, we can join the linkings before and after that monopole since it doesn't change the charge. Without loss of generality we will disregard the a solution in the following discussion.

By induction the i th condition reads then

$$n(1 - 2\#b + 2\#\tilde{b})(N'_i - N_i) + 2N_iN'_i = 0 \quad (66)$$

with the restriction that $\#b - \#\tilde{b}$ takes only the values 0 and 1. Then we will obtain a chain of alternating solutions $b\tilde{b} \dots b\tilde{b}b\tilde{b} \dots$ as the only possible solution. Some diagrams are presented in figure 3.

The above results are consistent with global charge conservation. And moreover they show that local charge violation is quite restricted. given some charge q in the first boundary it can change to \bar{q} and return to its previous form q again as many times as it wants but it can become nothing else.

Lets go on to consider now 3 charges propagating from one boundary to the other (or equivalently evolving in time)

$$\begin{array}{lcl} Q_1 & \rightarrow & \bar{Q}_1 \\ Q_2 & \rightarrow & \bar{Q}_2 \\ Q_3 = -Q_1 - Q_2 & \rightarrow & \bar{Q}_3 = -\bar{Q}_1 - \bar{Q}_2 \end{array} \quad (67)$$

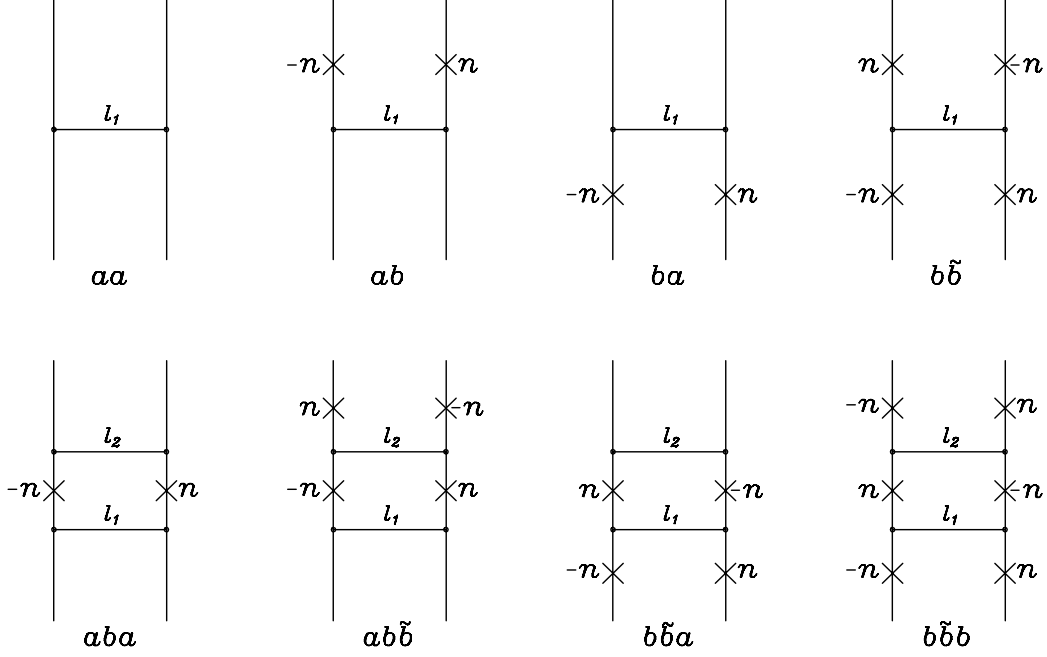


Figure 3: Some of the possible diagrams for two charge propagation, $i = 2$ and $i = 3$.

as pictured in figure 2. The phase is know

$$\begin{aligned}
\Phi_3 = \frac{2}{k} \Bigg\{ & Q_1 Q_2 l_0 - Q_2 (Q_1 + Q_2) d_0 - Q_1 (Q_1 + Q_2) k_0 + \\
& + \sum_{i=1}^{j-1} [\bar{Q}_{1(i)}^2 l_i - \bar{Q}_{1(i)} \bar{Q}_{3(i)} d_i - \bar{Q}_{2(i)} \bar{Q}_{3(i)} k_i] \\
& + \bar{Q}_1 \bar{Q}_2 l_j - \bar{Q}_2 (\bar{Q}_1 + \bar{Q}_2) d_j - \bar{Q}_1 (\bar{Q}_1 + \bar{Q}_2) k_j \Bigg\}
\end{aligned} \tag{68}$$

where we defined

$$\begin{aligned}
\bar{Q}_{1(i)} &= Q_1 + \frac{k}{2} \sum_{a=1}^i N_i & \bar{Q}_1 &= Q_1 + \frac{k}{2} N \\
\bar{Q}_{2(i)} &= Q_2 + \frac{k}{2} \sum_{a=1}^i M_i & \bar{Q}_2 &= Q_2 + \frac{k}{2} M \\
\bar{Q}_{3(i)} &= Q_1 + Q_2 - \frac{k}{2} \sum_{a=1}^i N'_i & \bar{Q}_3 &= Q_1 + Q_2 - \frac{k}{2} N'
\end{aligned} \tag{69}$$

Now we can choose to factor out some combination of l_0 , d_0 , k_0 , l_j , d_j and k_j . But it is now clear from the previous discussion of two charges propagation what factorizations we should look for. So we are only going to eliminate all the 0's, all the j 's or some combination of

0's and j 's for the three linkings. Defining the total linking numbers $l = l_0 + \sum l_i + l_j$, $d = d_0 + \sum d_i + d_j$ and $k = k_0 + \sum k_i + k_j$ we can replace the l_0 , d_0 and k_0 dependence and obtain the phases appearing in the correlation of three fields given in (22) for z_{12} , z_{23} and z_{13} on l , d and k respectively. Considering extra non compact gauge fields with charges p_1 , p_2 and p_3 and summing over the previous variables we obtain the conditions that the combinations of conformal dimensions in (22) must be integer. The conformal dimensions are to be read considering the extra non compact gauge fields and charges $\Delta = Q^2/k + p^2/k'$ and $\bar{\Delta} = \bar{Q}^2/k + p^2/k'$. From now on let us use this definition of conformal dimensions. As discussed in section 2 from the point of view of the boundary CFT's these factors must indeed be integer to insure that the three point OPE are single valued.

In terms of the individual vertices it is not so clear what these conditions mean. Without loss of generality and in order to clarify it let us replace the sums over l , d and k by the sums over three new variables l_+ , d_+ and k_+ such that $l = l_+ + k_+$, $d = d_+ + l_+$ and $k = k_+ + d_+$. Summing over these new variables one obtains the following conditions

$$2\Delta_1, 2\Delta_2, 2\Delta_3 \in \mathbb{Z} \quad (70)$$

In a similar way we may change l_j , d_j and k_j dependence by l_+ , d_+ and k_+ and upon summation obtain the respective conditions

$$2\bar{\Delta}_1, 2\bar{\Delta}_2, 2\bar{\Delta}_3 \in \mathbb{Z} \quad (71)$$

Or using the definition (51) for l_0^\pm , d_0^\pm and k_0^\pm and replacing l_0^+ , d_0^+ and k_0^+ dependence by l_+ , d_+ and k_+ in the same way as before, after performing the sums in these last variables, we get the conditions

$$\Delta_1 + \bar{\Delta}_1, \Delta_2 + \bar{\Delta}_2, \Delta_3 + \bar{\Delta}_3 \in \mathbb{Z} \quad (72)$$

Similarly to the previous discussion for two charge propagation these conditions turn out to be equivalent to the mass shell and spin conditions of string theory.

The fundamental differences arrive in the remaining conditions. Taking the first two cases (70) and (71), where we replaced the 0's and j 's linking number sums, and summing over (l_{+j}, d_{+j}, k_{+j}) and (l_{+0}, d_{+0}, k_{+0}) we obtain in both cases three conditions. The first two are redundant, they correspond to (46) which has been obtained from the two charges propagation, the third one is new

$$\frac{4}{k} (Q_1 Q_2 - \bar{Q}_1 \bar{Q}_2) \in \mathbb{Z} \quad (73)$$

Nevertheless the solutions for irrational k end up by being of the same kind as before. There are two solutions

$$\begin{cases} N &= 0 \\ M &= 0 \end{cases} \quad \begin{cases} N &= -n_1 \\ M &= -n_2 \end{cases} \quad (74)$$

For (72) case, where we replaced the sum over $+$'s linking number, we obtain, upon summation over $(l_{+0}^-, d_{+0}^-, k_{+0}^-)$, the conditions two conditions (53) which were obtained in the two charges case. The third one is again new

$$\frac{2}{k} (Q_1 Q_2 - \bar{Q}_1 \bar{Q}_2) \in \mathbb{Z} \quad (75)$$

The solutions for the full diagrams are very similar to the ones of two charges. With the charges oscillating simultaneous between Q_1 , Q_2 and $\bar{Q}_1 = Q_1 - n_1 k/2$, $\bar{Q}_2 = Q_2 - n_2 k/2$. For four charges there are no new conditions.

After all these algebra let us summarize what we obtained. We started with a theory which, at the perturbative level, has the allowed charges of the form $Q = m + (k/4)n$. We showed, under certain assumptions that, if the non-perturbative processes are taken into account, the charge spectrum is modified (restricted) quite drastically. The allowed left/right pair of charges are in one-to-one correspondence with the processes which have non-vanishing quantum amplitudes. All these processes can be organized in a lattice Γ with elements $l = (Q, \bar{Q} = Q + (k/2)N)$ with N either 0 or $-n$. Moreover a Lorentzian product \circ of signature $(+, -)$ emerge naturally from (46), (53) and (75) defined as

$$l \circ l' = \frac{2}{k} (QQ' - \bar{Q}\bar{Q}') \quad (76)$$

And we can go even further and extract very important properties of the lattice, it is **even** due to (53)

$$l \circ l = \frac{2}{k} (Q^2 - \bar{Q}^2) \in 2\mathbb{Z} \quad (77)$$

and **integer** due to (75).

$$l \circ l' = \frac{2}{k} (QQ' - \bar{Q}\bar{Q}') \in \mathbb{Z} \quad (78)$$

Note that (73) is redundant since it is necessarily obeyed if (75) is.

By inspection, for $N = -n$ the lattice is **self-dual** as well. But not for $N = 0$. In the next section we explain how to exclude $N = 0$ case and obtain what we seek. The lattice is then exactly the one for the bosonic string theory with one compact dimension. It is the Narain lattice for compactification in S^1 . For toroidal compactification of several dimensions is enough to consider several compact $U(1)$. This situation is well described in section 6.

4 Boundary conditions

We still need to understand how we can select between $N = 0$ and $N = -n$ cases. The key is to consider different combinations of the boundary conditions introduced in (6). Further,

to obtain a CFT in the boundary (whether it is chiral or not) as an effective 2D theory of the 3D TMGT we have to identify the two boundaries Σ_L and Σ_R in some way. These are the mechanisms that allow us to build several string theories out of the same 3D theory. Note that this constructions are only possible due to the Maxwell term in the action and the existence of monopoles.

We learned in the last section that the bulk theory only allows the charge $q = m + kn/4$ to become $\bar{q} = m - kn/4$ or keep it to be the same charge q . What is actually n ? Let us go back to (16), $2\pi n$ is the flux of the magnetic field. So upon some kind of boundary identification our theory only admits the ones that hold one of the conditions

$$\begin{aligned} N = 0 \quad & \int_{\Sigma_L} d^2 z_L B = \int_{\Sigma_R} d^2 z_R B \\ N = -n \quad & \int_{\Sigma_L} d^2 z_L B = - \int_{\Sigma_R} d^2 z_R B \end{aligned} \tag{79}$$

Let us consider a map (coordinate transformation) that transforms a vector in Σ_R into another vector in Σ_L , giving us the identification *rule*. There will be two kinds of maps. One which maintain the relative orientation of both 2d boundaries, let us call it parallel ($//$), and the other kind that reverses the relative orientation, let us call it perpendicular (\perp). The names are chosen by the relative identification of the axes from boundary to boundary as pictured in figure 4.

Note that the induced $\tilde{\epsilon}$ does not change from boundary to boundary. From the 3d point of view we are reversing time in one boundary and swaping the complex space coordinates. From the point of view of the bulk nothing special is happening.

Let us define the difference of fluxes as

$$\delta\phi = \int_R B - \int_L B = \int_{\Sigma_R} d^2 z \epsilon^{ij} \partial_i A_j - \int_{\Sigma_L} d^2 z \epsilon^{ij} \partial_i A_j \tag{80}$$

Given our boundary conditions we can evaluate this difference explicitly. Let us write what the magnetic field is for each of the allowed boundary conditions

$$\begin{aligned} C. \quad \phi_C &= - \int \partial_{\bar{z}} A_z \\ \bar{C}. \quad \phi_{\bar{C}} &= \int \partial_z A_{\bar{z}} \end{aligned} \tag{81}$$

In these equations we took the magnetic field definition $B = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z$ and imposed the respective boundary conditions. Taking into account the boundary identifications and labeling the different combinations of boundary conditions accordingly we compute the flux

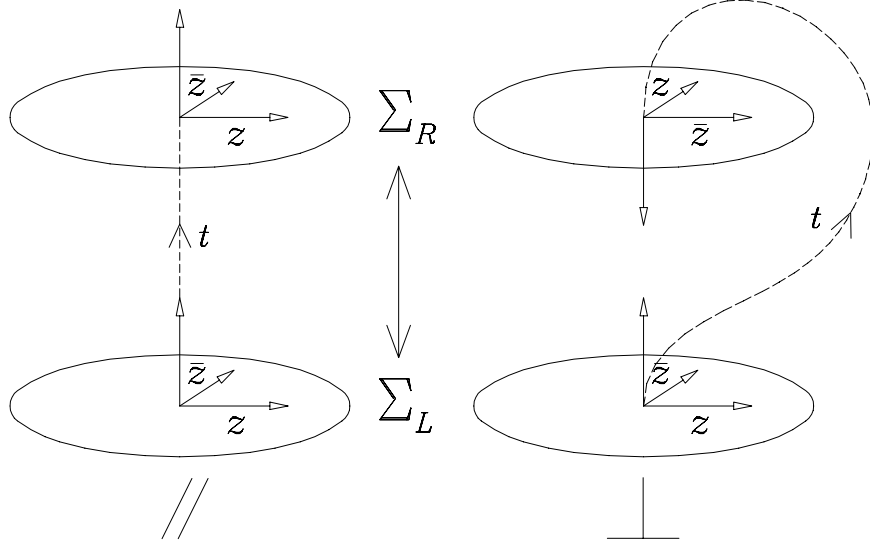


Figure 4: Boundary identification with the same orientation (parallel; //) and reversed orientation (perpendicular; \perp)

difference

$$\begin{aligned}
CC_{//}. \quad & \delta\phi = 0 \\
CC_{\perp}. \quad & \delta\phi = -\int(\partial_{\bar{z}}A_z + \partial_zA_{\bar{z}}) = -2\pi(2n) \\
C\bar{C}_{//}. \quad & \delta\phi = -\int(\partial_{\bar{z}}A_z + \partial_zA_{\bar{z}}) = -2\pi(2n) \\
C\bar{C}_{\perp}. \quad & \delta\phi = 0
\end{aligned} \tag{82}$$

Note that for parallel type of identifications the fields and integrals are summed without any relative change. For the perpendicular type we have to change the space indices of the fields to their conjugates and the measure in the integral changes sign (in one boundary, say the right one). The results are summarized in figure 5.

So the conclusion is that $Q = \bar{Q}$ corresponds to $CC_{//}$, $\bar{C}\bar{C}_{//}$, CC_{\perp} or $\bar{C}\bar{C}_{\perp}$ and $Q - \bar{Q} = (k/2)n$ to CC_{\perp} , $\bar{C}\bar{C}_{\perp}$, $C\bar{C}_{//}$ or $\bar{C}C_{//}$ boundary conditions. Note that in principle there may exist other choices of the maps which would give different boundary theories. We leave this to a future work.

Note that the spectrum for NC or $N\bar{C}$ boundary conditions must be obtained by truncating the spectrum of CC_{\perp} or $C\bar{C}_{//}$. We simply pick $\bar{Q} = 0$ for the N boundary, for instance

$$m = \frac{k}{4}n \tag{83}$$

and the charges in the other boundary become

$$Q = \frac{k}{2}n \tag{84}$$

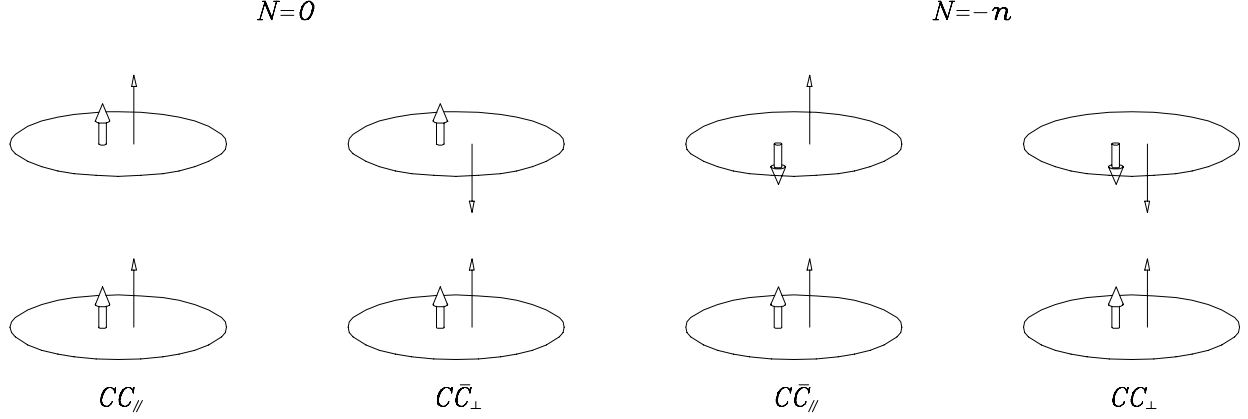


Figure 5: Boundary identifications for several combinations of boundary conditions. The fat arrow represents magnetic flux and the thin one boundary orientation.

Of course for irrational k the only solution for the condition (83) is $m = n = 0$. We end up with a very poor and empty theory in the boundary. In section 6 it will be clarified in which cases condition (83) allows some dynamics in the boundary. Trying to truncate the spectrum of $CC_{//}$ or $C\bar{C}_{\perp}$ we will set straight away $Q = \bar{Q} = 0$ since the charges are equal in both boundaries killing the hope of finding any dynamics in the boundaries.

We choose from now on to work with CC_{\perp} or $C\bar{C}_{//}$ type of boundary conditions since they are the ones which give us the desired spectrum in the boundaries CFT's. Further, as we just explained, N boundary conditions can easily be obtained from them.

5 RCFT's and Fusion Rules

A CFT is rational when its infinite set of primary fields (vertex operators) can be organized in a finite number N of families usually called *primary blocks*. In each of those blocks we can choose one minimal field that is the generator of that family. There is an algebra between these families, or if we want, between the N minimal fields called the fusion algebra. The fusion rules define this second algebra of fields.

In what follows we are only going to discuss the holomorphic part of the $c=1$ RCFT of a bosonic field ϕ living in a circle of radius $R = \sqrt{2p'/q}$. Here p' and q are integers. The vertex operators are

$$V_{Q_c} = \exp(2\pi i Q_c \phi) \quad Q_c = \frac{r}{R} + s \frac{R}{2} \quad r, s \in \mathbb{Z} \quad (85)$$

Where Q_c are the charges (or momenta) of the theory. The conformal dimensions of these vertex operators are $\Delta = Q_c^2/2$. There are $N = 2p'q$ primary blocks (or families). The

generators V_{Q_λ} of such families are chosen such that their conformal dimensions are the lowest allowed by the theory. In terms of their charges these are

$$Q_\lambda = \frac{\lambda}{\sqrt{N}} \quad \lambda = 0, 1, 2, \dots, N-2, N-1 \quad (86)$$

We are going to call them from now on primary charges. In this way Q_λ runs from 0 to $\sqrt{N} - 1/\sqrt{N}$. The remaining fields of the theory are obtained by successive products of the generators. The charges in a family λ are

$$Q_{\lambda,L} = Q_\lambda + L\sqrt{N} \quad L \in \mathbb{Z} \quad (87)$$

The generators form the fusion algebra given by the fusion rules

$$Q_\lambda + Q_{\lambda'} = Q_{(\lambda+\lambda') \bmod N} \quad (88)$$

We took the liberty to express it in terms of the charges. Formally it is expressed in terms of fields or vertex operators $V_\lambda \times V_{\lambda'} = V_{(\lambda+\lambda') \bmod N}$. This simply means we pick two primary charges out of two families and add them. We will obtain a charge from a different family, but not necessarily the primary one. The fusion rules pick the primary charge of that new family. For further details on these subjects see [23].

For our normalization we have the following relations

$$k = 2R^2 \quad Q = RQ_c \quad (89)$$

In what follows we are going to explain how all this emerges with some naturalness from the bulk theory. Take then $k = 2p/q$ to be a rational number. Note that in the previous discussion $2p' = p$, so take even p . By inspection we check that the (m, n) space is divided in diagonal bricks containing $p \times q$ charges. They are distributed in diagonal layers of identically valued bricks as symbolically pictured in figure 6. To check this *diagonal structure* explicitly take a generic charge labeled by the pair (m, n) , it can be represented by any other pair $(m - pn'/2, n + qn')$

$$Q = m + \frac{p}{2q}n = m - \frac{p}{2}n' + \frac{p}{2q}(n + qn') \quad (90)$$

This simply represents a diagonal translation of n' bricks in the figure.

Note that this choice of the brick shape is not unique, we could take some other choice as long as it has dimension pq (e.g. parallelogram of sides p and q) and the following results would hold nevertheless. The reason for this particular choice is in order to get a direct

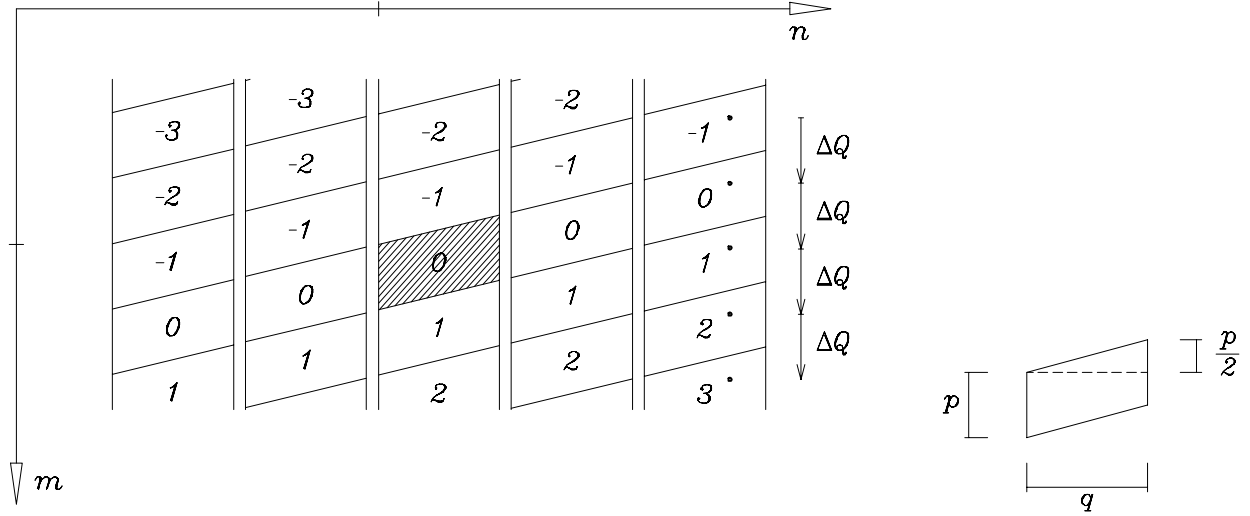


Figure 6: (m, n) structure for even p . The dashed region is the primary brick (r, s) , some examples are presented in figure 7. The numbers inside the blocks represent $L\Delta Q$ shift of the charge values as given by (95). Blocks with the same L have the same charge entries. In each brick there is exactly one element of level L belonging to each family.

parallel with the usual result of RCFT's. Further we consider one of such bricks to be the primary brick. It corresponds to the minimal charges (chosen to be positive) allowed by the theory built out of the lowest pairs of integers (r, s)

$$Q_{rs} = r + \frac{p}{2q}s = \frac{\lambda}{q} \quad \lambda = rq + \frac{p}{2}s = 0, 1, 2, \dots, pq - 2, pq - 1 \quad (91)$$

Examples for some values of k are given in figure 7. The geometric rule to organize the charges inside the brick is to order them in ascending order by their distance to the upper diagonal line of slope $k/4$.

So far we just reproduced the well know charge structure of RCFT's, it is now time to return to the bulk theory and justify it. Take again condition (53), replacing k , it reads now

$$\frac{p}{2q} (nN - N^2) \in \mathbb{Z} \quad (92)$$

which can be reexpressed as

$$N(n - N) = 0 \bmod q \quad (93)$$

The solutions for (93) can be easily computed to be

$$\begin{aligned} N &= 0 \bmod q \\ N &= -n + 0 \bmod q \end{aligned} \quad (94)$$

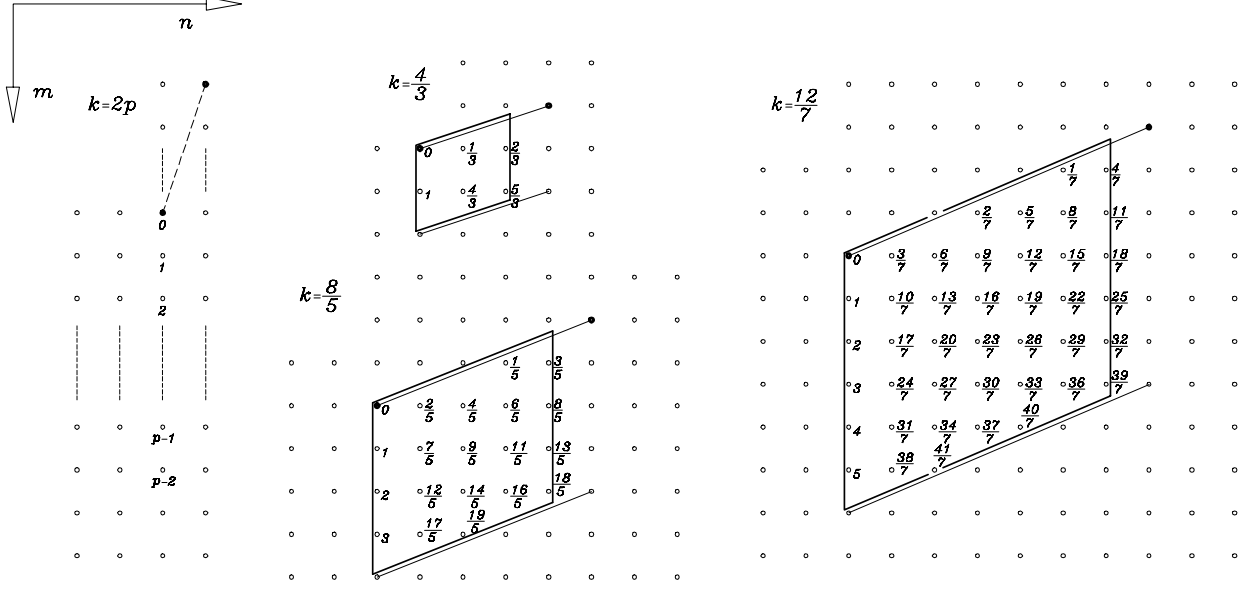


Figure 7: Primary charges distribution in the (m, n) plane, they constitute the primary brick (r, s) . The rule to order the charges is by their distance to the (thin) line connecting the charges $Q = 0$ (filled dots) of slope $k/4 = p/2q$.

which are equivalent to (46) and (47). There is one important lesson to take from this result, besides the previous allowed monopole-instanton process $n \rightarrow -n$ which is charge dependent, there is a new charge independent one:

$$\Delta Q = \pm p \quad (95)$$

This is actually the physical process that spans each of the families! To obtain the charges of some family in terms of m and n we can think on shift either $m \rightarrow m + pL$ or $n \rightarrow n + 2qL$. This is due to an infinite degeneracy in the (m, n) plane of the charges values expressed by (90). Figure 8 presents the distribution of charges for $k = 2p$. In this simplified case the families are simply organized along the m axis for even values of $2n$. This structure is repeated by shifts on the m axis of value p .

For odd p all the structure is similar but with bricks of dimension $p \times 2q$, take again a generic charge Q

$$Q = \frac{p n}{2q} + m = \frac{p (n + 2 n')}{2q} + m_p - p n' \quad (96)$$

Because p is odd, same valued charges correspond to pairs related by $(m, n) \rightarrow (m - pn', n + 2qn')$. The slope of the brick is the same $k/4$ but they are twice bigger in the n direction. All the rest works in the same way. This would correspond to a boson field living on a circle of radius $R = \sqrt{p/q}$.

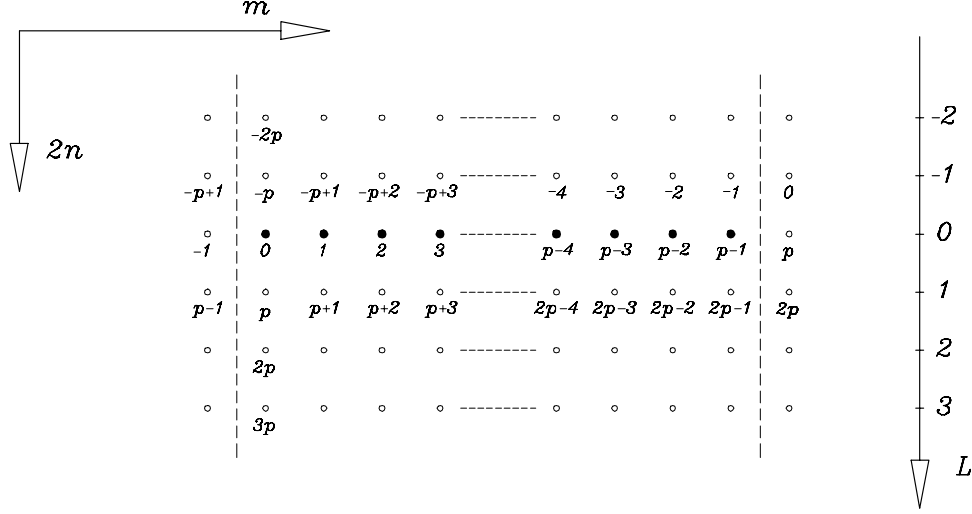


Figure 8: Charge structure for $k = 2p$. Only even values are considered in the n axis

6 Lattices and Heterotic String

Considering an action of the type (1) with compactified $U(1)^D$ gauge group (replacing k by a generic tensor K_{IJ}) we have

$$S = \int_{\mathcal{M}} d^2z dt \left[-\frac{\sqrt{-g(3)}}{4\gamma} F_I^{\mu\nu} F_{\mu\nu}^I + \frac{G_{IJ} + B_{IJ}}{8\pi} \epsilon^{\mu\nu\lambda} A_\mu^I \partial_\nu A_\lambda^J \right] \quad (97)$$

G_{IJ} and B_{IJ} stand for the symmetric and antisymmetric parts of K_{IJ} . Using the same procedure outline in section 2 we build the monopole-instanton which induces a charge change of

$$\Delta Q_I = \frac{G_{IJ}}{2} N^J \quad (98)$$

Using the Schrödinger picture we get the charge spectrum

$$Q_I = m_I + \frac{K_{IJ}}{4} n^J \quad (99)$$

Note that the antisymmetric part B_{IJ} is present in the charge but not in the monopole effects. This is due to the fact that it is manifested only at the level of the boundary. Note that in the action the term corresponding to B_{IJ} is a total derivative and can be completely integrated out to the boundary

$$\int_M B_{IJ} \epsilon^{\mu\nu\lambda} A_\mu^I \partial_\nu A_\lambda^J = \frac{1}{2} \left[- \int_{\Sigma_L} B_{IJ} \tilde{\epsilon}^{ij} A_i^I A_j^J + \int_{\Sigma_R} B_{IJ} \tilde{\epsilon}^{ij} A_i^I A_j^J \right] \quad (100)$$

Let us analyze first the case for $B_{IJ} = 0$. The previous discussion of section 4 follows in the same fashion. Choosing CC_\perp or CC_\parallel boundary conditions we have the desired relative

spectrum in each boundary, that is, $\mathbf{Q} = \mathbf{m} + K\mathbf{n}/4$ and $\bar{\mathbf{Q}} = \mathbf{m} - K\mathbf{n}/4$. This means that every monopole contribution of the form (98) has exactly $N^J = -n^J$.

What about if $B_{IJ} \neq 0$? Let us return to our boundary identifications, For CC_\perp the measure of the integrals in opposite boundaries change their relative sign, say $\int_R d^2z \rightarrow -\int_R d^2z$, due to the $2d$ measure changing sign. The fields in one boundary (say Σ_R) are swapped, $A_i \leftrightarrow A_j$. Note that we are not changing neither B_{IJ} nor $\tilde{\epsilon}^{ij}$ in the right integral, they are induced from the bulk and do not change by the boundary identifications. Thus this transformation has no effect in the B_{IJ} term

$$\int_{\Sigma_L} B_{IJ} \epsilon^{ij} A_i^I A_j^J \rightarrow \int_{\Sigma_L} B_{IJ} \epsilon^{ij} A_i^I A_j^J \quad (101)$$

For $CC_{//}$ nothing changes either. This means that B_{IJ} term does not change sign under any of our boundary identifications.

So we obtain the left/right spectrum

$$\begin{aligned} Q_I &= \frac{G_{IJ} + B_{IJ}}{4} n^J + m_I \\ \bar{Q}_I &= \frac{-G_{IJ} + B_{IJ}}{4} n^J + m_I \end{aligned} \quad (102)$$

such that the charge difference $Q_I - \bar{Q}_I = G_{IJ} n^J / 2$ is indeed the monopole contribution.

We obtain a lattice $l = (\mathbf{Q}, \bar{\mathbf{Q}})$ with the Lorentzian product of signature $(+, -)$ defined as

$$l \circ l' = 2G^{(-1)IJ} (Q_I Q'_J - \bar{Q}_I \bar{Q}'_J) \quad (103)$$

where G^{-1} stands for the inverse of G_{IJ} . The signature of the product has D plus and D minus. The properties of this lattice are the same as the ones for the previous $D = 1$ case and follow in a similar way from the bulk theory as presented in section 3. The lattice is integer, even and self-dual.

Let us analyze which lattices do exist for CN boundary conditions. As explained before (section 4) they can be obtained by truncating the lattices where $N^J = -n^J$. Imposing then CN is equivalent to truncating the lattice by choosing $\bar{\mathbf{Q}} = 0$. Similarly to (83), this means we are selecting elements of our lattice such that

$$m_I = \frac{G_{IJ} - B_{IJ}}{4} n^J \quad (104)$$

This sublattice has elements $l = (\mathbf{Q}, 0)$ with \mathbf{Q} built out of \mathbf{n} and \mathbf{m} obeying (104). Once $\bar{\mathbf{Q}} = 0$ the Lorentzian product (103) becomes, for this particular sublattice, simply $2G^{IJ} Q_I Q_J$. So it becomes Euclidean for this particular choice of elements. But the

properties of being even, integer and self dual are inherited from the full lattice. It is a known fact that the only even integer self dual Euclidean lattices are of dimension $0 \bmod 8$. Of dimension 8 there is only one, Γ^8 , the root lattice of E_8 . From (104) we can derive the spectrum of allowed \mathbf{Q} 's

$$Q_I = \frac{G_{IJ}}{2} n^J \quad (105)$$

Note that the resulting spectrum is independent of B_{IJ} . Due to the spectrum (105) and by closure of the lattice under the defined product (103) we are obliged not to restrict \mathbf{n} . Anyhow, as stated before, the charge lattice must be the root lattice of E_8 , it is clear that the only choice that holds this result is $G = 2C$. C is the standard Cartan matrix of this group (the diagonal elements have the value 2 and the off diagonal -1). The corresponding Dynkin diagram is pictured in figure 9. Let us return to (104). It must be a realizable condition for every \mathbf{n} , Thus we have no choice but to impose $G_{IJ} - B_{IJ} = 0 \bmod 4$.

We are left with

$$\begin{aligned} G_{IJ} &= 2C_{IJ} \\ B_{IJ} &= (2 + 4r)C_{IJ} \quad J > I, r \in \mathbb{Z} \end{aligned} \quad (106)$$

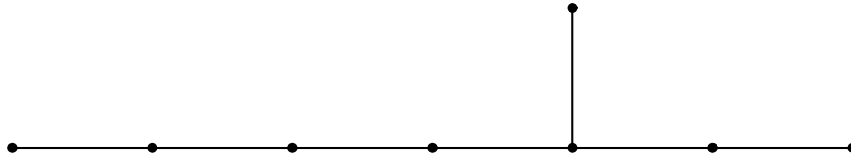


Figure 9: Dynkin Diagram of E_8

There are two lattices of dimension 16 obeying these properties: $\Gamma^8 \times \Gamma^8$ with elements in the root lattice of $E_8 \times E_8$ and Γ^{16} . We retrieve the well known results of string theory. Note that in the first works [9] of non interacting Heterotic string there is no mention to the tensor B_{IJ} (the free spectrum depends solely on G_{IJ}). Only after the introduction of interactions and an effective action for the theory [10] it becomes clear that there is a need for B_{IJ} . For $r = 0$ we retrieve the result stated in [24] and references there in up to normalization of $1/4$ on K_{IJ} .

7 Conclusions

Most of the constraints on string theories exist because of their intimate relations to $2D$ geometry, $2D$ conformal field theories and their symmetries. One of these constraints is the modular invariance which puts severe restrictions on the spectrum of the theories. Modular invariance, as employed by Narain in the context of toroidal compactification, reduces the

physical problem of finding the allowed momenta and winding modes in string theory to the problem of constructing lattices on a pseudo-Euclidean space \mathbb{R}^{d_R, d_L} on which physical invariants are modular invariant functions. These lattices turn out to be *self-dual*, *integer* and *even* lattices.

In this paper we have reproduced the results of Narain compactification starting from 3D gauge dynamics without referring to modular invariance at all. The fact that 3D topologically massive gauge theory carries full information about the lattices of string theory is quite fascinating. Our approach and the modular invariance construction are compatible and moreover as we will comment on it below they are logically connected. But there is still an element of surprise here because of the crucial difference between the ideology of these two approaches. To see this let us summarize what we have obtained. We started with a TMGT defined on a three manifold with two disconnected boundaries. Gauge degrees of freedom became dynamical on the boundaries generating chiral CFT's. Each boundary of the three manifold is interpreted as a “chiral worldsheet” of string theory, meaning that the left and right modes are physically separated. This is the frame work of Topological Membrane theory. We have considered the compact Abelian theory which has a discrete charge spectrum of the form $\mathbf{Q} = \mathbf{m} + k\mathbf{n}/4$. A particle with charge \mathbf{Q} is inserted at one boundary and it travels through the bulk interacts with all possible charge violating instantons on its path and links with other charges and emerges as a new charge $\bar{\mathbf{Q}} = \mathbf{m} - k\mathbf{n}/4$ on the other boundary. The path of the charged particle is represented by a Wilson line in the bulk theory. After taking care of all the linkings and instanton interactions it is quite an amusing result to obtain that the emerging charge in the other boundary is of the $\bar{\mathbf{Q}}$ form and nothing else. The Aharonov-Bohm phases of the linkings interfere in a way that for a particle which does not have the charge $\pm\bar{\mathbf{Q}}$ there is zero probability to emerge in the other boundary. Moreover a natural self-dual Lorentzian lattice structure emerges from the linkings of the Wilson lines as it was shown in Section 4. The connection between our approach and the modular invariance arguments was not worked out in this paper but naively one can see that modular transformations in the boundaries will yield linkings of the Wilson lines in the bulk. This is why we have gotten the same results.

There are a couple of important things which need to be addressed in future. One of them is the bulk interpretation of A-D-E classification [25] of modular invariant partition functions in the boundary. An other important issue is to study non-orientable surfaces which appear for non-oriented strings. We hope to address these issues in forthcoming publications.

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